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Math 253

Honors Project: Volume of n-sphere

Volume of Sphere in 2-Space

We know the volume of a sphere in 2-space (or the area of a circle) can be calculated through Calculus 2 methods, namely trig-substitution. We have a circle of radius R centered at the origin, and integrate the half circle above the $x-$axis, $y = \sqrt{R^2 - x^2}$, from $-R$ to R.

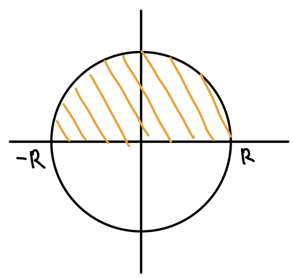

Figure 1: Area of Half Circle

$$Area = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx$$

Because of the symmetry, we can change the bounds to 0 to R and multiply the integral by 2:

$$= 4\int_{0}^{R} \sqrt{R^2 - x^2}\,dx$$

Using trig-sub:

$$x = R\sin\theta$$
$$dx = R\cos\theta d\theta$$

$$= 4\int_{0}^{\frac{\pi}{2}} \sqrt{R^2 - R^2\sin^2\theta}R\cos\theta\,d\theta$$
$$= 4\int_{0}^{\frac{\pi}{2}} R^2\cos^2\theta\,d\theta$$
$$= 4R^2\int_{0}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$
$$= 4R^2\int_{0}^{\frac{\pi}{2}} \frac{1}{2}[1+\cos(2\theta)]\,d\theta$$
$$= 2R^2\int_{0}^{\frac{\pi}{2}} 1+\cos(2\theta)\,d\theta$$

$$= 2R^2[\theta + \frac{1}{2}\sin(2\theta)]\Big|_0^{\frac{\pi}{2}}$$
$$= 2R^2[\frac{\pi}{2} + \frac{1}{2}\sin(\pi)]$$
$$= 2R^2(\frac{\pi}{2})$$
$$= \boxed{\pi R^2}$$

Volume of Sphere in 3-Space

Moving into 3-space, the volume of a sphere with radius R can be thought of as a volume of revolution, using the disk method. We are revolving the half circle, $y = \sqrt{R^2 - x^2}$, around the $x-$axis.

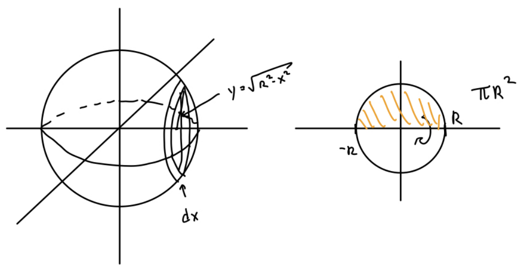

Figure 2: Disk Method for Sphere

$$Volume = \pi \int_{-R}^{R} (\sqrt{R^2 - x^2})^2 dx$$
$$= \pi \int_{-R}^{R} R^2 - x^2 dx$$
$$= \pi[R^2 x - \frac{1}{3}x^3]\Big|_{-R}^{R}$$
$$= \pi[R^3 - \frac{1}{3}R^3 - (-R^3 + \frac{1}{3}R^3)]$$
$$= \pi[2R^3 - \frac{2}{3}R^3]$$
$$= \boxed{\frac{4}{3}\pi R^3}$$

Generalizing the Volume Formula

Before we try to generalize the volume formula of a sphere, let's go over the simple case of a dimension 1 sphere. This would essentially be a line, from $-R$ to R whose "*volume*" would be $2R$. This can be calculated with the following:

$$"Volume" = \int_{-R}^{R} dx = x\Big|_{-R}^{R} = R + R = \boxed{2R}$$

Now that we have the volume formula for the first three dimensions, let's put them into a table and see what we have so far.

Dimension (n)	**Area to Integrate**	**Integral**	**Volume:** $V(R)$
1	Δx	$\int_{-R}^{R} dx$	2R
2	$2\sqrt{R^2 - x^2}\Delta x$	$2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx$	πR^2
3	$\pi(R^2 - x^2)\Delta x$	$\pi \int_{-R}^{R} (\sqrt{R^2 - x^2})^2 dx$	$\frac{4}{3}\pi R^3$

There are a few patterns we notice by looking at this table.

- The formula for volume always has a R^n term in it.
- The coefficients that are being multiplied with the R^n term shows up in the "Area to Integrate" for the next dimension.
- For each increase in dimension, the $(R^2 - x^2)$ term gets increased by a power of $\frac{1}{2}$.

With this in mind, we can construct the integral for the volume of a sphere for the 4th dimension. We take the $\frac{4}{3}\pi$ from the volume formula of dimension 3 and increase the power of $(R^2 - x^2)$ by $\frac{1}{2}$ to get the following integral:

$$Volume = \frac{4}{3}\pi \int_{-R}^{R} (R^2 - x^2)^{\frac{3}{2}} dx$$

This integral can be solved using trig-substitution and trig identities. Solving this gives:

$$\frac{4}{3}\pi \int_{-R}^{R} (R^2 - x^2)^{\frac{3}{2}} dx = \boxed{\frac{1}{2}\pi^2 R^4}$$

We can then take this and obtain the formula for the volume of a sphere in the 5th dimension. The integral for the 5th dimension looks like:

$$\frac{1}{2}\pi^2 \int_{-R}^{R} (R^2 - x^2)^2 dx$$

This integral can easily be solved by expanding the polynomial. Solving this integral gives:

$$\begin{aligned}
\frac{1}{2}\pi^2 \int_{-R}^{R} (R^2 - x^2)^2 dx &= \frac{1}{2}\pi^2 \int_{-R}^{R} (R^4 - 2R^2x^2 + x^4) dx \\
&= \frac{1}{2}\pi^2 [R^4 x - \frac{2}{3}R^2 x^3 + \frac{1}{5}x^5]\Big|_{-R}^{R} \\
&= \boxed{\frac{8}{15}\pi^2 R^5}
\end{aligned}$$

Lets now put the $n = 4$ and $n = 5$ dimensions integral and volume formula into our table:

Dimension (n)	**Area to Integrate**	**Integral**	**Volume: $V(R)$**
1	Δx	$\int_{-R}^{R} dx$	2R
2	$2\sqrt{R^2 - x^2}\Delta x$	$2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx$	πR^2
3	$\pi(R^2 - x^2)\Delta x$	$\pi \int_{-R}^{R}(\sqrt{R^2 - x^2})^2 dx$	$\frac{4}{3}\pi R^3$
4	$\frac{4}{3}\pi(R^2 - x^2)^{\frac{3}{2}}\Delta x$	$\frac{4}{3}\pi \int_{-R}^{R}(R^2 - x^2)^{\frac{3}{2}} dx$	$\frac{1}{2}\pi^2 R^4$
5	$\frac{1}{2}\pi^2(R^2 - x^2)^2\Delta x$	$\frac{1}{2}\pi^2 \int_{-R}^{R}(R^2 - x^2)^2 dx$	$\frac{8}{15}\pi^2 R^5$

We can make a new table with the different terms that we see reocurring in the volume formula. Let's further split this into two tables, one table for the odd dimensions and one table for the even dimensions.

Table 1: Odd Dimension

Dimension (n)	**Volume: $V(R)$**	**Coefficient**	π	R
1	2R	2	π^0	R^1
3	$\frac{4}{3}\pi R^3$	$\frac{4}{3}$	π^1	R^3
5	$\frac{8}{15}\pi^2 R^5$	$\frac{8}{15}$	π^2	R^5

Table 2: Even Dimension

Dimension (n)	**Volume: $V(R)$**	**Coefficient**	π	R
2	πR^2	1	π^1	R^2
4	$\frac{1}{2}\pi^2 R^4$	$\frac{1}{2}$	π^2	R^4

From these tables we can see more clearly a pattern for the various terms in the volume formula for an $n-$dimensional sphere. Lets first look at the easier pieces, π and R. We can see that the R term is always just R to the $n-$th power, R^n. For odd n, the π term is raised to the $\frac{n-1}{2}$, giving $\pi^{\frac{n-1}{2}}$. For even n, the π term is raised to the $\frac{n}{2}$, giving $\pi^{\frac{n}{2}}$.
Before we look at the coefficients, lets look the Gamma function. It might not seem obvious at first, but it is a key piece in deriving the formula for the volume of a sphere in the $n-$ dimension.

The Gamma Function

The definition of the gamma function is:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x > 0$$

The gamma function is related to taking a factorial. The definition of a factorial is:

$$x! = x(x-1)(x-2)(x-3)\cdots \times 2 \times 1$$

The gamma function can also be written as $(x-1)!$. We can equation the two with the following:

$$x! = x(x-1)! = \Gamma(x+1)$$

Two important identity (which we will include but not prove) of the gamma function which are the following where:

$$\Gamma(x+1) = (x)\Gamma(x)$$
$$\Gamma(1) = 1$$

This identity becomes useful very soon. First, lets find $\Gamma(\frac{1}{2})$. Using the definition of the gamma function:

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt$$
$$= \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt$$

Using u-sub with:

$$u = t^{\frac{1}{2}} \& u^2 = t$$
$$du = \tfrac{1}{2}t^{-\frac{1}{2}}dt$$
$$dt = 2t^{\frac{1}{2}}du$$

$$= \int_0^\infty t^{-\frac{1}{2}}e^{-u^2}(2t^{\frac{1}{2}})du$$
$$= 2\int_0^\infty e^{-u^2}du$$

From previous work, we know that:

$$\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$$

Since e^{-x^2} is an even function:

$$2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$$
$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

We plug this back into the step where we left off for $\Gamma(\frac{1}{2})$ and get:

$$2\int_0^\infty e^{-u^2}du = 2(\frac{\sqrt{\pi}}{2})$$
$$\Gamma(\frac{1}{2}) = \boxed{\sqrt{\pi}}$$

An important piece to finding the formula for volume of an $n-$dimensional sphere is the term $(\frac{n}{2})!$. The gamma function comes into play when we are generalizing the expansion of this. Let's start with finding the factorial of $\frac{1}{2}$, or $(\frac{1}{2})!$. We will combine all of the previous definitions to generalize the expansion of $(\frac{n}{2})!$.

$$(x)! = \Gamma(x+1) = x\Gamma(x)$$

Now we let $x = \frac{n}{2}$:

$$(\frac{n}{2})! = \Gamma(\frac{n}{2}+1) = \frac{n}{2}\Gamma(\frac{n}{2})$$

Let $n = 1$:

$$(\frac{1}{2})! = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$$

Now, let us go ahead and keep following the pattern inductively.

$$(\frac{1}{2})! = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$$

$$(\frac{2}{2})! = \Gamma(\frac{2}{2}+1) = \frac{2}{2}\Gamma(\frac{2}{2}) = \Gamma(1) = 1$$

$$(\frac{3}{2})! = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}(\frac{1}{2})! = \frac{3}{2}(\frac{\pi}{2}) = \frac{3}{4}\sqrt{\pi}$$

$$(\frac{4}{2})! = \Gamma(\frac{4}{2}+1) = \frac{4}{2}\Gamma(\frac{4}{2}) = \frac{4}{2}(1)! = 2$$

$$(\frac{5}{2})! = \Gamma(\frac{5}{2}+1) = \frac{5}{2}\Gamma(\frac{5}{2}) = \frac{5}{2}(\frac{3}{2})! = \frac{5}{2}(\frac{3}{4}\sqrt{\pi}) = \frac{15}{8}\sqrt{\pi}$$

$$(\frac{6}{2})! = \Gamma(\frac{6}{2}+1) = \frac{6}{2}\Gamma(\frac{6}{2}) = 3(2)! = 6$$

$$(\frac{7}{2})! = \Gamma(\frac{7}{2}+1) = \frac{7}{2}\Gamma(\frac{7}{2}) = \frac{7}{2}(\frac{5}{2})! = \frac{7}{2}(\frac{15}{8}\sqrt{\pi}) = \frac{105}{16}\sqrt{\pi}$$

Relating the Gamma Function to the Coefficients in Volume Formula

We can see a relationship between the values we get from $(\frac{n}{2})!$ and the coefficients we had earlier. Lets put them into a table and notice the relation:

Dimension (n)	**Coefficient**	$(\frac{n}{2})!$
1	2	$\frac{1}{2}\sqrt{\pi}$
2	1	1
3	$\frac{4}{3}$	$\frac{3}{4}\sqrt{\pi}$
4	$\frac{1}{2}$	2
5	$\frac{8}{15}$	$\frac{15}{8}\sqrt{\pi}$

We can see that the coefficient is a result of taking the inverse of $(\frac{n}{2})!$. For odd n, we need to compensate with a $\sqrt{\pi}$.

Dimension (n)	**Coefficient**	$(\frac{n}{2})!$	$\frac{1}{(\frac{n}{2})!}$
1	2	$\frac{1}{2}\sqrt{\pi}$	$\frac{2}{\sqrt{\pi}}$
2	1	1	1
3	$\frac{4}{3}$	$\frac{3}{4}\sqrt{\pi}$	$\frac{4}{3\sqrt{\pi}}$
4	$\frac{1}{2}$	2	$\frac{1}{2}$
5	$\frac{8}{15}$	$\frac{15}{8}\sqrt{\pi}$	$\frac{8}{15\sqrt{\pi}}$

Now, we can incorporate this into the table we made earlier with the odd- and even- dimensions.

Table 3: Odd Dimension

Dimension (n)	**Volume:** $V(R)$	**Coefficient**	π	R
1	2R	2	π^0	R^1
3	$\frac{4}{3}\pi R^3$	$\frac{4}{3}$	π^1	R^3
5	$\frac{8}{15}\pi^2 R^5$	$\frac{8}{15}$	π^2	R^5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{\sqrt{\pi}}{(\frac{n}{2})!}\pi^{\frac{n-1}{2}}R^n$	$\frac{\sqrt{\pi}}{(\frac{n}{2})!}$	$\pi^{\frac{n-1}{2}}$	R^n

Table 4: Even Dimension

Dimension (n)	**Volume:** $V(R)$	**Coefficient**	π	R
2	πR^2	1	π^1	R^2
4	$\frac{1}{2}\pi^2 R^4$	$\frac{1}{2}$	π^2	R^4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{(\frac{n}{2})!}\pi^{\frac{n}{2}} R^n$	$\frac{1}{(\frac{n}{2})!}$	$\pi^{\frac{n}{2}}$	R^n

Formula for Volume of Sphere of Even and Odd n

We have the formula to acquire the volume of a sphere in an odd- and even- dimension.

For even n, we get:

$$V_n(R) = \frac{1}{(\frac{n}{2})!}\pi^{\frac{n}{2}} R^n$$

For odd n, we get:

$$V_n(R) = \frac{\sqrt{\pi}}{(\frac{n}{2})!}\pi^{\frac{n-1}{2}} R^n$$

We can further simplify the odd n formula by combining the π terms:

$$\sqrt{\pi} \times \pi^{\frac{n-1}{2}} = \pi^{\frac{1}{2}+\frac{n-1}{2}} = \pi^{\frac{n}{2}}$$

Now the equation for odd n and even n are the same, giving the formula for the volume of a $n-$dimensional sphere:

$$V_n(R) = \frac{1}{(\frac{n}{2})!}\pi^{\frac{n}{2}} R^n$$

Simplifying the Formula

We can simplify this formula a bit further using the following:

$$(\frac{n}{2})! = \frac{n}{2}\Gamma\frac{n}{2} = \Gamma(\frac{n}{2} + 1)$$

Now, we use can swap $(\frac{n}{2})!$ with $\Gamma(\frac{n}{2} + 1)$ to get:

$$\boxed{V_n(R) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} R^n}$$